Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 23, 24

Lecturer: Hunter Stufflebeam

Textbook Sections: 11.9

Todays Topic: Algebra and Calculus with Power Series

Warm up: Let's get into the swing of things by revisiting an old friend:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad x \in (-1,1). \qquad (*)$$

In the last lecture, we introduced the point of view of treating an infinite series as a function. The "converse thought process" of finding an infinite series formula for a given function is also extremely useful, and is a POV we'll emphasize today.

Examples: Find a power series representation of the following:

1) $f(x) = \frac{1}{1+x^2}$

Use $(*)$. $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

By the Ratio test, the interval of convergence is $(-1,1)$ (or note $|x^2|<1$ is necessary from what we know about geometric series).

2) $f(x) = \frac{x^2}{x+2}$

Again use $(*)$. $\frac{x^2}{x+2} = \frac{x^2}{2}\frac{1}{1-(-x/2)} = \frac{x^2}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{x^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n$

From our geometric series knowledge, this series formula is convergent when $|x/2|<1$, i.e. $|x|<2$.

Soon, we'll introduce a way to produce such power series representations for more complicated functions. But first....

Q: Can we use the methods of calculus on these power series?

A: Inside the domain of convergence, we can differentiate and integrate a power series term-by-term, just like we do for finite sums of functions like polynomials. This is a nontrivial result, because power series are infinite sums. More precisely, the nontrivial fact is that the limit defining the infinite series can be switched with the limit defining the derivative/integral.

Theorem: (Term-by-term Diff'n and Int'n)

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$.

Then

(1) $f(x)$ is differentiable on $(a-R, a+R)$, where

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x-a)^n\right) = \sum_{n=0}^{\infty} c_n \cdot n (x-a)^{n-1} = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$$
$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

This series representation of $f'(x)$ also has radius of convergence R.

(2) $f(x)$ has an antiderivative on $(a-R, a+R)$:

$$F(x) = \int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$
$$= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots$$

This series representation of $F(x)$ also has radius of convergence R.

⚠ The behavior of these series representations might differ from the behavior of $f(x)$'s series representation at the endpoints $x = a \pm R$.

N.B. Returning to the remark before the Theorem, we see that the proof should boil down to showing that

(1) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \sum_{n=0}^{\infty} c_n \frac{(x+h-a)^n - (x-a)^n}{h} = \lim_{h\to 0} \lim_{N\to\infty} \sum_{n=0}^{N} c_n \frac{(x+h-a)^n-(x-a)^n}{h}$$

$$= \lim_{N\to\infty} \lim_{h\to 0} \sum_{n=0}^{N} c_n \frac{(x+h-a)^n-(x-a)^n}{h} = \lim_{N\to\infty} \sum_{n=0}^{N} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

(2) Once we justify (1), then (2) follows by differentiating the proposed series for $F(x)$ and employing (1)!

Key Point: Now we know how to differentiate and integrate important functions like the Bessel functions eg. $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. This is super important for applications, where we often understand series representations better than the "functions themselves outright!" Doing calculus operations on power series also proves very useful in eg) differential equations.

Examples:

(1) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. Then

- $J_0'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{2n}{(n!)^2} x^{2n-1}$
- $\int J_0(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \frac{1}{2n+1} x^{2n+1}$.

(2) Express $f(x) = \frac{1}{(1-x)^2}$ as a power series.

Notice that $f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$

(3) Find a power series representation of $\ln(1+x)$, and determine where it converges.

Note that $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. So, $\ln(1+x)$ is an antiderivative of $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$. Thus, we have (for some $C \in \mathbb{R}$)

$$\ln(1+x) = \int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

Since we need $0 = \ln(1) = \ln(1+0) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (0)^{n+1} = C$,

we conclude that $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$.

By the ratio test, $R = 1$ and the domain of convergence is $(0, 2)$.

(4) Evaluate $\int \frac{1}{1+x^7}\,dx$.

Try to do this using our earlier integration techniques—I dare you!

It's actually much easier to approach this problem using power series:

1) Note that $\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n}$.

2) Note that this converges for $|x|<1$.

3) Use the theorem to integrate term-by-term:

$$\int \frac{1}{1+x^7}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} x^{7n+1}$$

(5) Show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ solves the ODE $f'(x) = f(x)$.

In particular, show that $f(x) = e^x$.

1) Let's first determine the domain of convergence: Let $a_n = \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

Compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$

for every $x \in \mathbb{R}$.

Thus, the ratio test says $f(x)$ is well-defined for every $x \in \mathbb{R}$,

i.e. the domain of convergence is $(-\infty, \infty)$.

2) Use term-by-term differentiation:

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

3) Integrate! Formally: (assuming $f(x) \neq 0$ for all x, WLOG suppose $f > 0$)

$$f'(x) = f(x) \implies \frac{f'(x)}{f(x)} = 1$$

(change variable names for clarity)

$$\int_0^x \frac{f'(s)}{f(s)}\,ds \overset{u=f(s)}{=} \int_{f(0)}^{f(x)} \frac{1}{u}\,du = \int_1^{f(x)} \frac{1}{u}\,du = \ln f(x)$$

$$\int_0^x 1 \cdot ds = x \implies f(x) = e^x.$$

Check this formal computation: $\left(\frac{f(x)}{e^x}\right)' = \frac{e^x f'(x) - f(x)e^x}{e^{2x}} = e^{-x}\left(f'(x) - f(x)\right) = 0$

$\implies f(x) = c \cdot e^x$ for all x & $f(0) = 1 \implies c = 1$.

Recap of this week:

Pre-recs: Sequences and series of real numbers, and convergence tests.

Lecture 22: Power series introduction

Def: $\sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n \in \mathbb{R}$, $a \in \mathbb{R}$ are fixed, and x is a variable

Goal: Understand such series as *functions* of x.

Application: Series representations can make analysis of functions (eg finding antiderivatives, solving ODEs) much easier. Many functions studied in the sciences are also naturally given as series (eg Bessel functions). Series representations are also useful for doing numerical computations.

Key Fact: $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is a nice function when x is in a domain of convergence, i.e., $\exists R \geq 0$ (the radius of convergence) so that $f(x)$ converges when $|x-a| < R$ and diverges when $|x-a| > R$.

⚠ $f(x)$ could converge/diverge at $x = a \pm R$ (the endpoints of the domain)

Key Tools: To find R (when given a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$) we can try the ratio and root tests, computing

$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a|$ or $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|}\,|x-a|$.

and then finding $R > 0$ so that $|x-a| < R$ makes these limits < 1.

At the endpoints, we investigate using other tests.

Lecture 23, 24: Calculus on Power Series

Goal: When thinking of $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ as a function, we would like to use the techniques of calculus:

Key Result We can differentiate/integrate $f(x)$ by doing these operations "term-by-term" in the power series, preserving the radius of convergence.

Applications: Can represent functions as power series, and do calculus operations on these power series instead! This is often mechanically simpler.